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SBV REGULARITY FOR HAMILTON-JACOBI EQUATIONS IN $\mathbb{R}^n$

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Abstract. In this paper we study the regularity of viscosity solutions to the following Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R} \times \mathbb{R}^n\,.$$

In particular, under the assumption that the Hamiltonian $H \in C^2(\mathbb{R}^n)$ is uniformly convex, we prove that $D_x u$ and $\partial_t u$ belong to the class $SBV_{loc}(\Omega)$.

1. Introduction

In this paper, we consider viscosity solutions u to Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n. \tag{1}$$

As it is well known, solutions of the Cauchy problem for (1) develop singularities of the gradient in finite time, even if the initial data $u(0,\cdot)$ is extremely regular. The theory of viscosity solutions, introduced by Crandall and Lions 30 years ago, provides several powerful existence and uniqueness results which allow to go beyond the formation of singularities. Moreover, viscosity solutions are the limit of several smooth approximations of (1). For a review of the concept of viscosity solution and the related theory for equations of type (1) we refer to [4, 5, 11].

In this paper we are concerned about the regularity of such solutions, under the following key assumption:

$$H \in C^2(\mathbb{R}^n) \qquad \text{and} \qquad c_H^{-1} Id_n \le D^2 H \le c_H Id_n \quad \text{for some } c_H > 0. \tag{2}$$

There is a vast literature about this issue. As it is well-known, under the assumption (2), any viscosity solution u of (1) is locally semiconcave in x. More precisely, for every $K \subset\subset \Omega$ there is a constant C (depending on K, Ω and c_H) such that the function $x \mapsto u(t,x) - C|x|^2$ is concave on K. This easily implies that u is locally Lipschitz and that ∇u has locally bounded variation, i.e. that the distributional Hessian $D_x^2 u$ is a symmetric matrix of Radon measures. It is then not difficult to see that the same conclusion holds for $\partial_t D_x u$ and $\partial_{tt} u$. Note that this result is independent of the boundary values of u and can be regarded as an interior regularization effect of the equation.

The rough intuitive picture that one has in mind is therefore that of functions which are Lipschitz and whose gradient is piecewise smooth, undergoing jump discontinuities along a set of codimension 1 (in space and time). A refined regularity theory, which confirms this picture and goes beyond, analyzing the behavior of the functions where singularities are formed, is available under further assumptions on the boundary values of u (we refer to the book [5] for an account on this research topic). However, if the boundary values are just Lipschitz, these results do not apply and the corresponding viscosity solutions might be indeed quite rough, if we understand their regularity only in a pointwise sense.

In this paper we prove that the BV regularization effect is in fact more subtle and there is a measure-theoretic analog of "piecewise C^1 with jumps of the gradients". As a consequence of our analysis, we know for instance that the singular parts of the Radon measures $\partial_{x_i x_j} u$, $\partial_{x_i t} u$ and $\partial_{tt} u$ are concentrated on a rectifiable set of codimension 1. This set is indeed the measure theoretic jump set $J_{D_x u}$ of $D_x u$ (see below for the precise definition). This excludes, for instance, that the second derivative of u can have a complicated fractal behaviour. Using the language introduced in [8] we say that $D_x u$ and $\partial_t u$ are (locally) special functions of bounded variation, i.e. they belong to the space SBV_{loc} (we refer to the monograph [2] for more details). A typical example of a 1-dimensional function which belongs to BV but not to SBV is the classical Cantor staircase (cp. with Example 1.67 of [2]).

Theorem 1.1. *Let u be a viscosity solution of* (1)*, assume* (2) *and set $\Omega_t := \{x \in \mathbb{R}^n : (t,x) \in \Omega\}$. Then, the set of times*

$$S := \{t : D_x u(t,.) \notin SBV_{loc}(\Omega_t)\} \tag{3}$$

is at most countable. In particular $D_x u, \partial_t u \in SBV_{loc}(\Omega)$.

Corollary 1.2. *Under assumption* (2)*, the gradient of any viscosity solution u of*

$$H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{4}$$

belongs to $SBV_{loc}(\Omega)$.

Theorem 1.1 was proved first by Luigi Ambrosio and the second author in the special case $n = 1$ (see [3] and also [13] for the extension to Hamiltonians H depending on (t,x) and u). Some of the ideas of our proof originate indeed in the work [3]. However, in order to handle the higher dimensional case, some new ideas are needed. In particular, a key role is played by the geometrical theory of monotone functions developed by Alberti and Ambrosio in [1].

2. Preliminaries: the theory of monotone functions

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a continuous function $u : \Omega \to \mathbb{R}$ is* semiconcave *if, for any convex $K \subset\subset \Omega$, there exists $C_K > 0$ such that*

$$u(x+h) + u(x-h) - 2u(x) \le C_K |h|^2, \tag{5}$$

for all $x, h \in \mathbb{R}^n$ with $x, x-h, x+h \in K$. The smallest nonnegative costant C_K such that (5) *holds on K will be called* semiconcavity constant of u on K.

Next, we introduce the concept of superdifferential.

Definition 2.2. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. The set $\partial u(x)$, called the* superdifferential *of u at point $x \in \Omega$, is defined as*

$$\partial u(x) := \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y-x)}{|y-x|} \le 0 \right\}. \tag{6}$$

Using the above definition we can describe some properties of semiconcave functions (see Proposition 1.1.3 of [5]):

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ a compact convex set. Let $u : \Omega \to \mathbb{R}$ be a semiconcave function with semiconcavity constant $C_K \geq 0$. Then, the function*

$$\tilde{u} : x \mapsto u(x) - \frac{C_K}{2}|x|^2 \qquad \text{is concave in } K. \tag{7}$$

In particular, for any given $x, y \in K$, $p \in \partial\tilde{u}(x)$ and $q \in \partial\tilde{u}(y)$ we have that

$$\langle q - p, y - x\rangle \leq 0. \tag{8}$$

From now on, when u is a semi–concave function, we will denote the set-valued map $x \to \partial\tilde{u}(x) + C_K x$ as ∂u. An important observation is that, being $\tilde{u}$ concave, the map $x \to \partial\tilde{u}(x)$ is a maximal monotone function.

2.1. **Monotone functions in $\mathbb{R}^n$.** Following the work of Alberti and Ambrosio [1] we introduce here some results about the theory of monotone functions in $\mathbb{R}^n$. Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map (or multifunction), i.e. a map which maps every point $x \in \mathbb{R}^n$ into some set $B(x) \subset \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ we define:

- the *domain* of B, $Dm(B) := \{x : B(x) \neq \emptyset\}$,
- the *image* of B, $Im(B) := \{y : \exists x, y \in B(x)\}$,
- the *graph* of B, $\Gamma B := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in B(x)\}$,
- then *inverse* of B, $[B^{-1}](x) := \{y : x \in B(y)\}$.

Definition 2.4. *Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a multifunction, then*

(1) *B is a* monotone *function if*

$$\langle y_1 - y_2, x_1 - x_2\rangle \leq 0 \qquad \forall x_i \in \mathbb{R}^n, y_i \in B(x_i), i = 1, 2. \tag{9}$$

(2) *A monotone function B is called* maximal *when it is maximal with respect to the inclusion in the class of monotone functions, i.e. if the following implication holds:*

$$A(x) \supset B(x) \text{ for all } x, A \text{ monotone } \Rightarrow A = B. \tag{10}$$

Observe that in this work we assume $\leq$ in (9) instead of the most common $\geq$. However, one can pass from one convention to the other by simply considering $-B$ instead of B. The observation of the previous subsection is then summarized in the following Theorem.

Theorem 2.5. *The supergradient ∂u of a concave function is a maximal monotone function.*

An important tool of the theory of maximal monotone functions, which will play a key role in this paper, is the Hille-Yosida approximation (see Chapters 6 and 7 of [1]):

Definition 2.6. *For every $\varepsilon > 0$ we set $\Psi_\varepsilon(x, y) := (x - \varepsilon y, y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and for every maximal monotone function B we define B_ε as the multifunction whose graph is $\Psi_\varepsilon(\Gamma B)$, that is, $\Gamma B_\varepsilon = \{(x - \varepsilon y, y) : (x, y) \in \Gamma B\}$. Hence*

$$B_\varepsilon := (\varepsilon Id - B^{-1})^{-1}. \tag{11}$$

In the next Theorems we collect some properties of maximal monotone functions B and their approximations B_ε defined above.

Theorem 2.7. *Let B be a maximal monotone function. Then, the set $S(B) := \{x : B(x)$ is not single valued$\}$ is a $\mathcal{H}^{n-1}$ rectifiable set. Let $\tilde{B} : Dm(B) \to \mathbb{R}^n$ be such that $\tilde{B}(x) \in \tilde{B}(x)$ for every x. Then $\tilde{B}$ is a measurable function and $B(x) = \{\tilde{B}(x)\}$ for a.e. x. If $Dm(B)$ is open, then $D\tilde{B}$ is a measure, i.e. $\tilde{B}$ is a function of locally bounded variation.*

If K_i is a sequence of compact sets contained in the interior of $Dm(B)$ with $K_i \downarrow K$, then $B(K_i) \to B(K)$ in the Hausdorff sense. Therefore, the map $\tilde{B}$ is continuous at every $x \notin S(B)$.

Finally, if $Dm(B)$ is open and $B = \partial u$ for some concave function $u : Dm(B) \to \mathbb{R}$, then $\tilde{B}(x) = Du(x)$ for a.e. x (recall that u is locally Lipschitz, and hence the distributional derivative of u coincides a.e. with the classical differential).

Proof. First of all, note that, by Theorem 2.2 of [1], $S(B)$ is the union of rectifiable sets of Hausdorff dimension $n-k$, $k \geq 1$. This guarantees the existence of the classical measurable function $\tilde{B}$. The BV regularity when $Dm(B)$ is open is shown in Proposition 5.1 of [1].

Next, let K be a compact set contained in the interior of $Dm(B)$. By Corollary 1.3(3) of [1], $B(K)$ is bounded. Thus, since $\Gamma B \cap K \times \mathbb{R}^n$s is closed by maximal monotonicity, it turns out that it is also compact. The continuity claimed in the second paragraph of the Theorem is then a simple consequence of this observation.

The final paragraph of the Theorem is proved in Theorem 7.11 of [1]. $\square$

In this paper, since we will always consider monotone functions that are the supergradients of some concave functions, we will use ∂u for the supergradient and Du for the distributional gradient. A corollary of Theorem 2.7 is that

Corollary 2.8. *If $u : \Omega \to \mathbb{R}$ is semiconcave, then $\partial u(x) = \{Du(x)\}$ for a.e. x, and at any point where ∂u is single–valued, Du is continuous. Moreover $D^2 u$ is a symmetric matrix of Radon measures.*

Next we state the following important convergence theorem. For the notion of current and the corresponding convergence properties we refer to the work of Alberti and Ambrosio. However, we remark that very little of the theory of currents is needed in this paper: what we actually need is a simple corollary of the convergence in (ii), which is stated and proved in Subsection 5.2. In (iii) we follow the usual convention of denoting by $|\mu|$ the total variation of a (real-, resp. matrix-, vector- valued) measure μ. The theorem stated below is in fact contained in Theorem 6.2 of [1].

Theorem 2.9. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and let B be a maximal monotone function such that $\Omega \subset Dm(B)$. Let B_ε be the approximations given in Definition 2.6. Then, the following properties hold.*

(i) *B_ε is a $1/\varepsilon$-Lipschitz maximal monotone function on $\mathbb{R}^n$ for every $\varepsilon > 0$. Moreover, if $B = Du$, then $B_\varepsilon = Du_\varepsilon$ for the concave function*
$$u_\varepsilon(x) := \inf_{y \in \mathbb{R}^n} \left\{ u(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\} \tag{12}$$
(ii) *ΓB and ΓB_ε have a natural structure as integer rectifiable currents, and $\Gamma B_\varepsilon \llcorner \Omega \times \mathbb{R}^n$ converges to $\Gamma B \llcorner \Omega \times \mathbb{R}^n$ in the sense of currents as $\varepsilon \downarrow 0$.*

(iii) *$DB_\varepsilon \rightharpoonup^* D\tilde{B}$ and $|DB_\varepsilon| \rightharpoonup^* |D\tilde{B}|$ in the sense of measures on Ω.*

2.2. **BV and SBV functions.** We conclude the section by introducing the basic notations related to the space SBV (for a complete survey on this topic we address the reader to [2]). If $B \in BV(A, R^k)$, then it is possible to split the measure DB into three mutually singular parts:

$$DB = D_a B + D_j B + D_c B.$$

$D_a B$ denotes the absolutely continuous part (with respect to the Lebesgue measure). $D_j B$ denotes the jump part of DB. When A is a 1-dimensional domain, $D_j B$ consists of a countable sum of weighted Dirac masses, and hence it is also called the atomic part of DB. In higher dimensional domains, $D_j B$ is concentrated on a rectifiable set of codimension 1, which corresponds to the measure-theoretic jump set J_B of B. $D_c B$ is called the Cantor part of the gradient and it is the "diffused part" of the singular measure $D_s B := D_j B + D_c B$. Indeed

$$D_c B(E) = 0 \qquad \text{for any Borel set } E \text{ with } \mathcal{H}^{n-1}(E) < \infty. \tag{13}$$

For all these statements we refer to Section 3.9 of [2].

Definition 2.10. *Let $B \in BV(\Omega)$, then B is a special function of bounded variation, and we write $B \in SBV(\Omega)$, if $D_c B = 0$, i.e. if the measure DB has no Cantor part. The more general space $SBV_{loc}(\Omega)$ is defined in the obvious way.*

In what follows, when u is a (semi)-concave function, we will denote by $D^2 u$ the distributional hessian of u. Since Du is, in this case, a BV map, the discussion above applies. In this case we will use the notation $D^2_a u$, $D^2_j u$ and $D^2_c u$. An important property of $D^2_c u$ is the following regularity property.

Proposition 2.11. *Let u be a (semi)-concave function. If D denotes the set of points where ∂u is not single–valued, then $|D^2_c u|(D) = 0$.*

Proof. By Theorem 2.7, the set D is $\mathcal{H}^{n-1}$-rectifiable. This means in particular, that it is $\mathcal{H}^{n-1} - \sigma$ finite. By the property (13) we conclude $D^2_c u(E) = 0$ for every Borel subset E of D. Therefore $|D^2_c u|(D) = 0$. □

3. Hamilton-Jacobi equations

In this section we collect some definitions and well-known results about Hamilton-Jacobi equations. For a complete survey on this topic we redirect the reader to the vast literature. For an introduction to the topic we suggest the following sources [4],[5],[9]. In this paper we will consider the following Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0, \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n \ , \tag{14}$$

$$H(D_x u) = 0, \qquad \text{in } \Omega \subset \mathbb{R}^n \ , \tag{15}$$

under the assumption that

A1: The Hamiltonian $H \in C^2(\mathbb{R}^n)$ satisfies:

$$p \mapsto H(p) \text{ is convex and } \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

Note that this assumption is obviously implied by (2).

We will often consider $\Omega = [0,T] \times \mathbb{R}^n$ in (14) and couple it with the initial condition

$$u(0,x) \;=\; u_0(x) \tag{16}$$

under the assumption that

A2: The initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous and bounded.

Definition 3.1 (Viscosity solution). *A bounded, uniformly continuous function u is called a* viscosity solution *of* (14) *(resp.* (15)*) provided that*

(1) *u is a* viscosity subsolution *of* (14) *(resp.* (15)*): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a maximum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0,x_0) + H(D_x v(t_0,x_0)) \le 0 \qquad \textit{(resp. } H(Dv(x_0)) \le 0\textit{);} \tag{17}$$

(2) *u is a* viscosity supersolution *of* (14) *(resp.* (15)*): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a minimum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0,x_0) + H(D_x v(t_0,x_0)) \ge 0 \qquad \textit{(resp. } H(Dv(x_0)) \ge 0\textit{).} \tag{18}$$

In addition, we say that u solves the Cauchy problem (14)-(16) *on $\Omega = [0,T] \times \mathbb{R}^n$ if* (16) *holds in the classical sense.*

Theorem 3.2 (The Hopf-Lax formula as viscosity solution). *The unique viscosity solution of the initial-value problem* (14)-(16) *is given by the Hopf-Lax formula*

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ u_0(y) + tL\left(\frac{x-y}{t}\right) \right\} \qquad (t > 0, x \in \mathbb{R}^n), \tag{19}$$

where L is the Legendre transform of H:

$$L(q) := \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(p) \} \qquad (q \in \mathbb{R}^n). \tag{20}$$

In the next Proposition we collect some properties of the viscosity solution defined by the Hopf-Lax formula:

Proposition 3.3. *Let $u(t,x)$ be the viscosity solution of* (14)-(16) *and defined by* (19)*, then*

(i) ***A functional identity:*** *For each $x \in \mathbb{R}^n$ and $0 \le s < t \le T$, we have*

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ u(s,y) + (t-s)L\left(\frac{x-y}{t-s}\right) \right\}. \tag{21}$$

(ii) ***Semiconcavity of the solution:*** *For any fixed $\tau > 0$ there exists a constant $C(\tau)$ such that the function defined by*

$$u_t : \mathbb{R}^n \to \mathbb{R}^n \text{ with } u_t(x) := u(t,x), \tag{22}$$

is semiconcave with constant less than C for any $t \ge \tau$.

(iii) ***Characteristics:*** *The minimum point y in* (19) *is unique if and only if $\partial u_t(x)$ is single valued. Moreover, in this case we have $y = x - tDH(D_x u(t,x))$.*

(iv) ***The linear programming principle:*** *Let $t > s > 0$, $x \in \mathbb{R}^n$ and assume that y is a minimum for* (19)*. Let $z = \frac{s}{t}x + (1 - \frac{s}{t})y$. Then y is the* unique *minimum for $u_0(w) + sL((z-w)/s)$.*

Remark 3.4. *For a detailed proof of Theorem 3.2 and Proposition 3.3 we address the reader to Chapter 6 of* [5] *and Chapters 3, 10 of* [9] *.*

Next, we state a useful locality property of the solutions of (14).

Proposition 3.5. *Let u be a viscosity solution of* (14) *in Ω. Then u is locally Lipschitz. Moreover, for any $(t_0, x_0) \in \Omega$, there exists a neighborhood U of (t_0, x_0), a positive number δ and a Lipschitz function v_0 on $\mathbb{R}^n$ such that*

(Loc) *u coincides on U with the viscosity solution of*

$$\begin{cases} \partial_t v + H(D_x v) \;=\; 0 & \textit{in } [t_0-\delta, \infty[\times\mathbb{R}^n \\ \\ v(t_0-\delta, x) \;=\; v_0(x)\,. \end{cases} \tag{23}$$

This property of viscosity solutions of Hamilton-Jacobi equations is obviously related to the finite speed of propagation (which holds when the solution is Lipschitz) and it is well-known. One could prove it, for instance, suitably modifying the proof of Theorem 7 at page 132 of [9]. On the other hand we have not been able to find a complete reference for Proposition 3.5. Therefore, for the reader's convenience, we provide a reduction to some other properties clearly stated in the literature.

Proof. The local Lipschitz regularity of u follows from its local semiconcavity, for which we refer to [5]. As for the locality property (Loc), we let $\delta > 0$ and R be such that $C := [t_0-\delta, t_0+\delta] \times \overline{B}_R(x_0) \subset \Omega$. It is then known that the following dynamic programming principle holds for every $(t,x) \in C$ (see for instance Remark 3.1 of [6] or [10]):

$$u(t,x) \;=\; \inf\left\{ \int_\tau^t L(\dot\xi(s))\,ds + u(\tau, \xi(\tau)) \,\middle|\, \tau \le t, \xi \in W^{1,\infty}([\tau, t]), \right. \tag{24}$$

$$\left. \xi(t) = x \text{ and } (\tau, \xi(\tau)) \in \partial C \right\}.$$

The Lipschitz regularity of u and the convexity of L ensure that a minimizer exists. Moreover any minimizer is a straight line. Next, assume that $x \in B_\delta(x_0)$. If δ is much smaller than R, the Lipschitz regularity of u ensures that any minimizer ξ has the endpoint $(\tau, \xi(\tau))$ lying in $\{t_0-\delta\} \times B_R(x_0)$. Thus, for every $(t,x) \in [t_0-\delta, t_0+\delta] \times B_\delta(x_0)$ we get the formula

$$u(t,x) \;=\; \min_{y \in \overline{B}_R(x_0)} \left(u(t_0-\delta, y) + (t-t_0+\delta) L\left(\frac{x-y}{t-t_0+\delta}\right)\right). \tag{25}$$

Next, extend the map $\overline{B}_R(0) \ni x \mapsto u(t_0-\delta, x)$ to a bounded Lipschitz map $v_0 : \mathbb{R}^n \to \mathbb{R}$, keeping the same Lipschitz constant. Then the solution of (23) is given by the Hopf-Lax formula

$$v(t,x) \;=\; \min_{y \in \mathbb{R}^n} \left(v_0(y) + (t-t_0+\delta) L\left(\frac{x-y}{t-t_0+\delta}\right)\right). \tag{26}$$

If $(t,x) \in [t_0-\delta, t_0+\delta] \times B_\delta(0)$, then any minimum point y in (26) belongs to $\overline{B}_R(0)$, provided δ is sufficiently small (compared to R and the Lipschitz constant of v, which in turn is bounded independently of δ). Finally, since $v_0(y) = u(t_0-\delta, y)$ for every $y \in \overline{B}_R(0)$, (25) and (26) imply that u and v coincide on $[t_0-\delta, t_0+\delta] \times B_\delta(0)$ provided δ is sufficiently small. ◻

4. Proof of the main Theorem

4.1. **Preliminary remarks.** Let u be a viscosity solution of (14). By Proposition 3.5 and the time invariance of the equation, we can, without loss of generality, assume that u is a solution on $[0,T]\times\mathbb{R}^n$ of the Cauchy-Problem (14)-(16) under the assumptions A1, A2. Clearly, it suffices to show that, for every $j>0$, the set of times $S\cap]1/j,+\infty[$ is countable. Therefore, by Proposition 3.2 and the time–invariance of the Hamilton–Jacobi equations, we can restrict ourselves to the following case:

$$\exists C \text{ s.t. } u_\tau \text{ is semiconcave with constant less than } C \text{ and } |Du_\tau|\leq C\ \forall\tau\in[0,T]. \tag{27}$$

Arguing in the same way, we can further assume that

$$T \text{ is smaller than some constant } \varepsilon(C)>0, \tag{28}$$

where the choice of the constant $\varepsilon(C)$ will be specified later.

Next we consider a ball $B_R(0)\subset\mathbb{R}^n$ and a bounded convex set $\Omega\subset[0,T]\times\mathbb{R}^n$ with the properties that:

- $B_R(0)\times\{s\}\subset\Omega$ for every $s\in[0,T]$;
- For any $(t,x)\in\Omega$ and for any y reaching the minimum in the formulation (19), $(0,y)\in\Omega$ (and therefore the entire segment joining (t,x) to $(0,y)$ is contained in Ω).

Indeed, recalling that $\|Du\|_\infty<\infty$, it suffices to choose $\Omega:=\{(x,t)\in\mathbb{R}^n\times[0,T]:|x|\leq R+C'(T-t)\}$ where the costant C' is sufficiently large, depending only on $\|Du\|_\infty$ and H. Our goal is now to show the countability of the set S in (3).

4.2. **A function depending on time.** For any $s<t\in[0,T]$, we define the set–valued map

$$X_{t,s}(x)\ :=\ x-(t-s)DH(\partial u_t(x))\,. \tag{29}$$

Moreover, we will denote by $\chi_{t,s}$ the restriction of $X_{t,s}$ to the points where $X_{t,s}$ is single–valued. According to Theorem 2.7 and Proposition 3.3(iii), the domain of $\chi_{t,s}$ consists of those points where $Du_t(\cdot)$ is continuous, which are those where the minimum point y in (21) is unique. Moreover, in this case we have $\chi_{t,s}(x)=\{y\}$.

Clearly, $\chi_{t,s}$ is defined a.e. on Ω_t. With a slight abuse of notation we set

$$F(t)\ :=\ |\chi_{t,0}(\Omega_t)|\,, \tag{30}$$

meaning that, if we denote by U_t the set of points $x\in\Omega_t$ such that (19) has a unique minimum point, we have $F(t)=|X_{t,0}(U_t)|$.

The proof is then split in the following three lemmas:

Lemma 4.1. *The functional F is nonincreasing,*

$$F(\sigma)\geq F(\tau)\qquad \text{for any } \sigma,\tau\in[0,T] \text{ with } \sigma<\tau. \tag{31}$$

Lemma 4.2. *If ε in* (28) *is small enough, then the following holds. For any $t\in]0,T[$ and $\delta\in]0,T-t]$ there exists a Borel set $E\subset\Omega_t$ such that*

(i) *$|E|=0$, and $|D^2_c u_t|(\Omega_t\setminus E)=0$;*
(ii) *$X_{t,0}$ is single valued on E (i.e. $X_{t,0}(x)=\{\chi_{t,0}(x)\}$ for every $x\in E$);*
(iii) *and*

$$\chi_{t,0}(E)\cap\chi_{t+\delta,0}(\Omega_{t+\delta})=\emptyset. \tag{32}$$

Lemma 4.3. *If ε in (28) is small enough, then the following holds. For any $t \in]0,\varepsilon]$ and any Borel set $E \subset \Omega_t$, we have*

$$|X_{t,0}(E)| \geq c_0|E| - c_1 t \int_E d(\Delta u_t)\,, \tag{33}$$

where c_0 and c_1 are positive constants and Δu_t is the Laplacian of u_t.

4.3. **Proof of Theorem 1.1.** The three key lemmas stated above will be proved in the next two sections. We now show how to complete the proof of the Theorem. First of all, note that F is a bounded function. Since F is, by Lemma 4.1, a monotone function, its points of discontinuity are, at most, countable. We claim that, if $t \in]0,T[$ is such that $u_t \notin SBV_{loc}(\Omega_t)$, then F has a discontinuity at t.

Indeed, in this case we have

$$|D^2_c u_t|(\Omega_t) > 0. \tag{34}$$

Consider any $\delta > 0$ and let $B = E$ be the set of Lemma 4.2. Clearly, by Lemma 4.2(i) and (ii), (32) and (33),

$$F(t+\delta) \ \leq \ F(t) + c_1 t \int_E d\,\Delta_s u_t \ \leq \ F(t) + c_1 t \int_{\Omega_t} d\,\Delta_c u_t\,, \tag{35}$$

where the last inequality follows from $\Delta_s u_t = \Delta_c u_t + \Delta_j u_t$ and $\Delta_j u_t \leq 0$ (because of the semiconcavity of u).

Next, consider the Radon–Nykodim decomposition $D^2_c u_t = M|D^2_c u_t|$, where M is a matrix–valued Borel function with $|M| = 1$. Since we are dealing with second derivatives, M is symmetric, and since u_t is semiconcave, $M \leq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $-M$. Then $1 = |M|^2 = \lambda_1^2 + \ldots + \lambda_n^2$ and $-TrM = \lambda_1 + \ldots + \lambda_n$. Since $\lambda_i \geq 0$, we easily get $-TrM \geq 1$. Therefore,

$$-\,\Delta_c u_t \ = \ -TrM|D^2_c u_t| \ \geq \ |D^2_c u_t|\,. \tag{36}$$

Hence

$$F(t+\delta) \ \overset{(35)+(36)}{\leq} \ F(t) - c_1 t |D^2_c u_t|(\Omega_t)\,.$$

Letting $\delta \downarrow 0$ we conclude

$$\limsup_{\delta\downarrow 0} F(t+\delta) \ < \ F(t)\,.$$

Therefore t is a point of discontinuity of F, which is the desired claim.

4.4. **Easy corollaries.** The conclusion that $D_x u \in SBV(\Omega)$ follows from the slicing theory of BV functions (see Theorem 3.108 of [2]). In order to prove the same property for $\partial_t u$ we apply the Volpert chain rule to $\partial_t u = -H(D_x u)$. According to Theorem 3.96 of [2], we conclude that $[\partial_{x_j t}]_c u = -\sum_i \partial_i H(D_x u)[\partial_{x_j x_i}]_c u = 0$ (because $[D^2_x]_c u = 0$) and $[\partial_{tt}]_c u = -\sum_i \partial_i H(D_x u)[\partial_{x_i t}]_c u = 0$ (because we just concluded $[D^2_{xt}]_c u = 0$).

As for Corollary 1.2, let u be a viscosity solution of (15) and set $\widetilde{u}(t,x) := u(x)$. Then $\tilde{u}$ is a viscosity solution of

$$\partial_t \widetilde{u} + H(D_x \widetilde{u}) = 0$$

in $\mathbb{R} \times \Omega$. By our main Theorem 1.1 the set of times for which $D_x \widetilde{u}(t,.) \notin SBV_{loc}(\Omega)$ is at most countable. Since $D_x \widetilde{u}(t,\cdot) = Du$, for every t, we conclude that $Du \in SBV_{loc}(\Omega)$.

Remark 4.4. *The special case of this Corollary for $\Omega \subset \mathbb{R}^2$ was already proved in* [3] *(see Corollary 1.4 therein). We note that the proof proposed in* [3] *was more complicated than the one above. This is due to the power of Theorem 1.1. In* [3] *the authors proved the 1–dimensional case of Theorem 1.1. The proof above reduces the 2–dimensional case of Corollary 1.2 to the $2+1$ case of Theorem 1.1. In* [3] *the 2-dimensional case of Corollary 1.2 was reduced to the $1+1$ case of Theorem 1.1: this reduction requires a subtler argument.*

5. Estimates

In this section we prove two important estimates. The first is the one in Lemma 4.3. The second is an estimate which will be useful in proving Lemma 4.2 and will be stated here.

Lemma 5.1. *If $\varepsilon(C)$ in* (28) *is sufficiently small, then the following holds. For any $t \in]0,T]$, any $\delta \in [0,t]$ and any Borel set $E \subset \Omega_t$ we have*

$$\Big|X_{t,\delta}(E)\Big| \geq \frac{(t-\delta)^n}{t^n}\Big|X_{t,0}(E)\Big| . \tag{37}$$

5.1. Injectivity.

In the proof of both lemmas, the following remark plays a fundamental role.

Proposition 5.2. *For any $C > 0$ there exists $\varepsilon(C) > 0$ with the following property. If v is a semiconcave function with constant less than C, then the map $x \mapsto x - tDH(\partial v)$ is injective for every $t \in [0, \varepsilon(C)]$.*

Here the injectivity of a set–valued map B is understood in the following natural way

$$x \neq y \qquad \Longrightarrow \qquad B(x) \cap B(y) = \emptyset .$$

Proof. We assume by contradiction that there exist $x_1, x_2 \in \Omega_t$ with $x_1 \neq x_2$ and such that:

$$[x_1 - tDH(\partial v(x_1))] \cap [x_2 - tDH(\partial v(x_2))] \neq \emptyset.$$

This means that there is a point y such that

$$\begin{cases} \frac{x_1-y}{t} \in DH(\partial v(x_1)), \\ \frac{x_2-y}{t} \in DH(\partial v(x_2)); \end{cases} \Rightarrow \begin{cases} DH^{-1}(\frac{x_1-y}{t}) \in \partial v(x_1), \\ DH^{-1}(\frac{x_2-y}{t}) \in \partial v(x_2). \end{cases} \tag{38}$$

By the semiconcavity of v we get:

$$M(x_1,x_2) := \Big\langle DH^{-1}\Big(\frac{x_1-y}{t}\Big) - DH^{-1}\Big(\frac{x_2-y}{t}\Big), x_1 - x_2\Big\rangle \leq C|x_1-x_2|^2. \tag{39}$$

On the other hand, $D(DH^{-1})(x) = (D^2H)^{-1}(DH^{-1}(x))$ (note that in this formula, DH^{-1} denotes the inverse of the map $x \mapsto DH(x)$, whereas $D^2H^{-1}(y)$ denotes the matrix A which is the inverse of the matrix $B := D^2H(y)$). Therefore $D(DH^{-1})(x)$ is a symmetric matrix, with $D(DH^{-1})(x) \geq c_H^{-1} Id_n$. It follows that

$$\begin{aligned} M(x_1,x_2) &= t\Big\langle DH^{-1}\Big(\frac{x_1-y}{t}\Big) - DH^{-1}\Big(\frac{x_2-y}{t}\Big), \frac{x_1-y}{t} - \frac{x_2-y}{t}\Big\rangle \geq \\ &\geq \frac{t}{2c_H}\Big|\frac{x_1-y}{t} - \frac{x_2-y}{t}\Big|^2 \geq \frac{1}{2tc_H}|x_1-x_2|^2 \geq \frac{1}{2\varepsilon c_H}|x_1-x_2|^2. \end{aligned} \tag{40}$$

But if $\varepsilon > 0$ is small enough, or more precisely if it is chosen to satisfy $2\varepsilon c_H < \frac{1}{C}$ the two inequalities (39) and (40) are in contradiction. □

5.2. **Approximation.** We next consider u as in the formulations of the two lemmas, and $t \in [0,T]$. Then the function $\tilde{v}(x) := u(x) - C|x|^2/2$ is concave. Consider the approximations B_η (with $\eta > 0$) of $\partial\tilde{v}$ given in Definition 2.6. By Theorem 2.9(i), $B_\eta = D\tilde{v}_\eta$ for some concave function $\tilde{v}_\eta$ with Lipschitz gradient. Consider therefore the function $v_\eta(x) = \tilde{v}_\eta(x) + C|x|^2/2$. The semiconcavity constant of v_η is not larger than C.

Therefore we can apply Proposition 5.2 and choose $\varepsilon(C)$ sufficiently small in such a way that the maps

$$x \;\mapsto\; A(x) = x - tDH(\partial u_t) \qquad x \;\mapsto\; A_\eta(x) = x - tDH(Dv_\eta) \tag{41}$$

are both injective. Consider next the following measures:

$$\mu_\eta(E) \;:=\; |(Id - tDH(Dv_\eta))(E)| \qquad \mu(E) \;:=\; |(Id - tDH(\partial u_t))(E)|\,. \tag{42}$$

These measures are well-defined because of the injectivity property proved in Proposition 5.2.

Now, according to Theorem 2.9, the graphs ΓDv_η and $\Gamma\partial u_t$ are both rectifiable currents and the first are converging, as $\eta \downarrow 0$, to the latter. We denote them, respectively, by T_η and T. Similarly, we can associate the rectifiable currents S and S_η to the graphs ΓA and ΓA_η of the maps in (41). Note that these graphs can be obtained by composing $\Gamma\partial u_t$ and ΓDv_η with the following global diffeomorphism of $\mathbb{R}^n$:

$$(x,y) \;\mapsto\; \Phi(x,y) = x - tDH(y)\,.$$

In the language of currents we then have $S_\eta = \Phi_\sharp T_\eta$ and $S = \Phi_\sharp T$. Therefore, $S_\eta \to S$ in the sense of currents.

We want to show that

$$\mu_\eta \;\rightharpoonup^*\; \mu\,. \tag{43}$$

First of all, note that S and S_η are rectifiable currents of multiplicity 1 supported on the rectifiable sets $\Gamma A = \Phi(\Gamma\partial u_t)$ and $\Gamma A_\eta = \Phi(\Gamma B_\eta) = \Phi(\Gamma Dv_\eta)$. Since B_η is a Lipschitz map, the approximate tangent plane π to S_η in (a.e.) point $(x, A_\eta(x))$ is spanned by the vectors $e_i + DA_\eta(x)\cdot e_i$ and hence oriented by the n-vector

$$\vec{v} := \frac{(e_1 + DA_\eta(x)\cdot e_1)\wedge\ldots\wedge(e_n + DA_\eta(x)\cdot e_n)}{|(e_1 + DA_\eta(x)\cdot e_1)\wedge\ldots\wedge(e_n + DA_\eta(x)\cdot e_n)|}\,.$$

Now, by the calculation of Proposition 5.2, it follows that $\det DA_\eta \geq 0$. Hence

$$\langle dy_1\wedge\ldots\wedge dy_n, \vec{v}\rangle \;\geq\; 0\,. \tag{44}$$

By the convergence $S_\eta \to S$, (44) holds for the tangent planes to S as well.

Next, consider a $\varphi \in C^\infty_c(\Omega_t)$. Since both ΓA and ΓA_η are bounded sets, consider a ball $B_R(0)$ such that $\operatorname{supp}(\Gamma A), \operatorname{supp}(\Gamma A_\eta) \subset \mathbb{R}^n \times B_R(0)$ and let $\chi \in C^\infty_c(\mathbb{R}^n)$ be a cut-off function with $\chi|_{B_R(0)} = 1$. Then, by standard calculations on currents, the injectivity property of Proposition 5.2 and (44) imply that

$$\int \varphi d\mu \;=\; \langle S, \varphi(x)\chi(y) dy_1\wedge\ldots\wedge dy_n\rangle, \tag{45}$$

$$\int \varphi d\mu_\eta \;=\; \langle S_\eta, \varphi(x)\chi(y) dy_1\wedge\ldots\wedge dy_n\rangle\,. \tag{46}$$

Therefore, since $S_\eta \to S$, we conclude that

$$\lim_{\eta\downarrow 0} \int \varphi d\mu_\eta \;=\; \int \varphi d\mu\,.$$

This shows (43).

5.3. **Proof of Lemma 5.1.** First of all we choose ε so small that the conclusions of Proposition 5.2 and those of Subsection 5.2 hold.

We consider therefore, the approximations v_η of Subsection 5.2, we define the measures μ and μ_η as in (42) and the measures $\hat\mu$ and $\hat\mu_\eta$ as

$$\hat\mu(E) \;:=\; |(Id-(t-\delta)DH(\partial u_t))(E)| \qquad \hat\mu_\eta(E) \;:=\; |(Id-(t-\delta)DH(Dv_\eta))(E)|\,. \tag{47}$$

By the same arguments as in Subsection 5.2, we necessarily have $\hat\mu_\eta \rightharpoonup^* \hat\mu$.

The conclusion of the Lemma can now be formulated as

$$\hat\mu \;\geq\; \frac{(t-\delta)^n}{t^n}\mu\,. \tag{48}$$

By the convergence of the measures μ_η and $\hat\mu_\eta$ to μ and $\hat\mu$, it suffices to show

$$\hat\mu_\eta \;\geq\; \frac{(t-\delta)^n}{t^n}\mu_\eta\,. \tag{49}$$

On the other hand, since the maps $x \mapsto x - tDH(Dv_\eta)$ and $x \mapsto x-(t-\delta)DH(Dv_\eta)$ are both injective and Lipschitz, we can use the area formula to write:

$$\hat\mu_\eta(E) \;=\; \int_E \det\Big(Id_n - (t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x)\Big)\,dx, \tag{50}$$

$$\mu_\eta(E) \;=\; \int_E \det\Big(Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x)\Big)\,dx \tag{51}$$

Therefore, if we set

$$\begin{aligned} M_1(x) &:= Id_n-(t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x)\\ M_2(x) &:= Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x)\ , \end{aligned}$$

the inequality (48) is equivalent to

$$\det M_1(x) \;\geq\; \frac{(t-\delta)^n}{t^n}\det M_2(x) \qquad \text{for a.e. } x. \tag{52}$$

Note next that

$$\begin{aligned} \det M_1(x) &= \det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1}-(t-\delta)D^2v_\eta(x)\Big)\\ \det M_2(x) &= \det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1}-tD^2v_\eta(x)\Big) \end{aligned}$$

Set $A(x) := [D^2H(Dv_\eta(x))]^{-1}$ and $B(x) = D^2v_\eta(x)$. Then it suffices to prove that:

$$\det(A(x)-(t-\delta)B(x)) \;\geq\; \frac{(t-\delta)^n}{t^n}\det(A(x)-tB(x))\,. \tag{53}$$

Note that

$$A-(t-\delta)B \;=\; \frac{\delta}{t}A + \frac{t-\delta}{t}(A-tB)\,.$$

By choosing ε sufficiently small (but only depending on c_H and C), we can assume that $A - tB$ is a positive semidefinite matrix. Since A is a positive definite matrix, we conclude

$$A - (t-\delta)B \ \geq \ \frac{t-\delta}{t}(A - tB)\,. \tag{54}$$

A standard argument in linear algebra shows that

$$\det(A - (t-\delta)B) \ \geq \ \frac{(t-\delta)^n}{t^n}\det(A - tB) \tag{55}$$

which concludes the proof. We include, for the reader convenience, a proof of (54) $\Longrightarrow$(55). It suffices to show that, if E and D are positive semidefinite matrices with $E \geq D$, then $\det E \geq \det D$. Without loss of generality, we can assume that E is in diagonal form, i.e. $E = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, and that $E > D$. Then each λ_i is positive. Define $G := \mathrm{diag}\,(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Then

$$\mathrm{Id}_n \ \geq \ G^{-1}DG^{-1} = \tilde{D}\,.$$

Our claim would follow if we can prove $1 \geq \det \tilde{D}$, that is, if we can prove the original claim for E and D in the special case where E is the identity matrix. But in this case we can diagonalize E and D at the same time. Therefore $D = \mathrm{diag}\,(\mu_1, \ldots, \mu_n)$. But, since $E \geq D \geq 0$, we have $0 \leq \mu_i \leq 1$ for each μ_i. Therefore

$$\det E \ = \ 1 \ \geq \ \Pi_i \mu_i \ = \ \det D\,.$$

5.4. **Proof of Lemma 4.3.** As in the proof above we will show the Lemma by approximation with the functions v_η. Once again we introduce the measures μ_η and μ of (42). Then, the conclusion of the Lemma can be formulated as

$$\mu \ \geq \ c_0\mathcal{L}^n - tc_1\Delta u_t\,. \tag{56}$$

Since $\Delta v_\eta \rightharpoonup^* \Delta u_t$ by Theorem 2.9(iii), it suffices to show

$$\mu_\eta \ \geq \ c_0\mathcal{L}^n - tc_1\Delta v_\eta\,. \tag{57}$$

Once again we can use the area formula to compute

$$\mu_\eta(E) \ = \ \int_E \det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1} - tD^2v_\eta(x)\Big)dx \tag{58}$$

Since $D^2H \geq c_H^{-1}Id_n$ and $[D^2H]^{-1} \geq c_H^{-1}Id_n$, we can estimate

$$\det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1} - tD^2v_\eta(x)\Big) \ \geq \ c_H^{-n}\det\left(\frac{1}{c_H}Id_n - tD^2v_\eta(x)\right) \tag{59}$$

arguing as in Subsection 5.3. If we choose ε so small that $0 < \varepsilon < \frac{1}{2c_HC}$, then $M(x) := \frac{1}{2c_H}Id_n - tD^2v_\eta(x)$ is positive semidefinite. Therefore

$$\det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1} - tD^2v_\eta(x)\Big) \ \geq \ c_H^{-n}\det\left(\frac{1}{2c_H}Id_n + M(x)\right)\,. \tag{60}$$

Diagonalizing $M(x) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, we can estimate

$$\begin{aligned} \det\left(\frac{1}{2c_H} Id_n + M(x)\right) &= \left(\frac{1}{2c_H}\right)^n \prod_{i=1}^n (1 + 2c_H\lambda_i) \;\geq\; \left(\frac{1}{2c_H}\right)^n (1 + 2c_H \mathrm{Tr}\, M(x)) \\ &= c_2 - c_3 t \Delta v_\eta(x)\,. \end{aligned} \tag{61}$$

Finally, by (58), (59), (60) and (61), we get

$$\mu_\eta(E) \;\geq\; \int_E (c_0 - c_1 t \Delta v_\eta(x))\, dx\,.$$

This concludes the proof.

6. Proofs of Lemma 4.1 and Lemma 4.2

6.1. **Proof of Lemma 4.1.** The claim follows from the following consideration:

$$\chi_{t,0}(\Omega_t) \subset \chi_{s,0}(\Omega_s) \qquad \text{for every } 0 \leq s \leq t \leq T. \tag{62}$$

Indeed, consider $y \in \chi_{t,0}(\Omega_t)$. Then there exists $x \in \Omega_t$ such that y is the unique minimum of (19). Consider $z := \frac{s}{t}x + \frac{t-s}{t}y$. Then $z \in \Omega_s$. Moreover, by Proposition 3.3(iv), y is the unique minimizer of $u_0(w) + sL((z-w)/s)$. Therefore $y = \chi_{s,0}(z) \in \chi_{s,0}(\Omega_s)$.

6.2. **Proof of Lemma 4.2.** First of all, by Proposition 2.11, we can select a Borel set E of measure 0 such that

- $\partial u_t(x)$ is single-valued for every $x \in E$;
- $|E| = 0$;
- $|D^2_c u_t|(\Omega_t \setminus E) = 0$.

If we assume that our statement were false, then there would exist a compact set $K \subset E$ such that

$$|D^2_c u_t|(K) \;>\; 0\,. \tag{63}$$

and $X_{t,0}(K) = \chi_{t,0}(K) \subset \chi_{t+\delta,0}(\Omega_{t+\delta})$. Therefore it turns out that $X_{t,0}(K) = \chi_{t+\delta,0}(\tilde{K}) = X_{t+\delta,0}(\tilde{K})$ for some Borel set $\tilde{K}$.

Now, consider $x \in \tilde{K}$ and let $y := \chi_{t+\delta,0}(x) \in X_{t+\delta,0}(\tilde{K})$ and $z := \chi_{t+\delta,t}(x)$. By Proposition 3.3(iv), y is the unique minimizer of $u_0(y) + tL((z-y)/t)$, i.e. $\chi_{t,0}(z) = y$.

Since $y \in \chi_{t,0}(K)$, there exists z' such that $\chi_{t,0}(z')$. On the other hand, by Proposition 5.2, provided ε has been chosen sufficiently small, $\chi_{t,0}$ is an injective map. Hence we necessarily have $z' = z$. This shows that

$$X_{t+\delta,t}(\tilde{K}) \subset K\,. \tag{64}$$

By Lemma 5.1,

$$|K| \;\geq\; |X_{t+\delta,t}(\tilde{K})| \;\geq\; \frac{\delta^n}{(t+\delta)^n}|X_{t+\delta,0}(\tilde{K})| \;=\; \frac{\delta^n}{(t+\delta)^n}|X_{t,0}(K)|\,. \tag{65}$$

Hence, by Lemma 4.3

$$|K| \;\geq\; c_0|K| - c_1 t \frac{\delta^n}{(t+\delta)^n}\int_K d\,\Delta u_t\,. \tag{66}$$

On the other hand, recall that $K \subset E$ and $|E| = 0$. Thus, $\int_K d\,\Delta_s u_t = \int_K d\,\Delta u_t \geq 0$. On the other hand $\Delta_s u_t \leq 0$ (by the semiconcavity of u). Thus we conclude that $\Delta_s u_t$, and hence also $\Delta_c u_t$, vanishes indentically on K. However, arguing as in Subsection 4.3, we can show $-\Delta_c u_t \geq |D^2_c u_t|$, and hence, recalling (63), $-\Delta_c u_t(K) > 0$. This is a contradiction and hence concludes the proof.

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